

# Complex supermanifolds with many unipotent automorphisms

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**Keywords:** complex supermanifold; nilpotent vector field; unipotent automorphism  
**MSC2010:** 58A50, 58H15, 54H15

## Abstract

An automorphism on a complex supermanifold  $\mathcal{M}$  is called unipotent if it reduces to the identity on the associated graded supermanifold  $gr(\mathcal{M})$ . These automorphisms are close to be complementary to those responsible for homogeneity of a supermanifold. In analogy, their study yields results on the classification of supermanifolds. Unipotent automorphisms are induced by even global degree increasing vector fields  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$ . Plenitude of unipotent automorphisms is understood as follows: the presheaf of common kernels of the operators  $[X, \cdot]$  for  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$ , on superderivations vanishes up to errors of a fixed degree  $t$  and higher. The isomorphy class of such strictly  $t$ -nildominated supermanifolds is determined up to errors of degree  $t$  and higher by  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  and  $gr(\mathcal{M})$ . An example shows that a strictly  $t$ -nildominated supermanifold can be non-split, deformed already in degrees lower than  $t$ .

A complex supermanifold  $\mathcal{M}$  is called homogeneous if it admits a transitive action of a Lie supergroup  $\mathcal{G}$ . Here transitivity is defined by two conditions: first the underlying action  $G \times M \rightarrow M$  is transitive, in particular the even vector fields cover all (even) directions of the underlying manifold. Secondly the induced odd vector fields on  $\mathcal{M}$  cover all odd directions, i.e. all fiber directions of the associated vector bundle  $E \rightarrow M$ . Complex homogeneous supermanifolds were studied in [On96] and considering special classes of underlying manifolds e.g. in [On07]. A classification statement is given in [Vi11]: a complex homogeneous supermanifold can be biholomorphically identified with a quotient  $\mathcal{G}/\mathcal{H}$  of complex Lie supergroups. Further a condition for the existence of splittings can be found in [Vi15].

In the present paper we consider the global even vector fields on a complex supermanifold  $\mathcal{M}$  that reduce on the associated graded supermanifold  $gr(\mathcal{M}) = (M, \mathcal{O}_{\Lambda E})$  to zero. These

fields are in some sense complementary to those studied on homogeneous supermanifolds. Their Lie algebra  $\mathcal{V}_{\mathcal{M},\bar{0}}^{(2)}$  is bijectively identified via the exponential map with the group  $H^0(M, G_E)$  of unipotent automorphisms of  $\mathcal{M}$ , i.e. automorphisms reducing to the identity on  $gr(\mathcal{M})$ .  $H^0(M, G_E)$  is a normal divisor in the group of automorphisms of  $\mathcal{M}$ , and in the case of a compact underlying manifold, it is a complex Lie subgroup of the automorphism Lie supergroup of  $\mathcal{M}$  (see [BK15]). The study of supermanifolds with many unipotent automorphisms yields the counterpart of the classification results on supermanifolds with no unipotent automorphisms in [Ka16]. In detail, we give a condition under which the equivalence class of a supermanifold is encoded in the nilpotent Lie group  $H^0(M, G_E)$  and the complex vector bundle  $E \rightarrow M$ .

We give suitable notions for the plenitude of vector fields in  $\mathcal{V}_{\mathcal{M},\bar{0}}^{(2)}$  and deduce results on the classification and the splitting problem of complex supermanifolds in the non-homogeneous case. Elements of  $\mathcal{V}_{\mathcal{M},\bar{0}}^{(2)}$  can hardly be considered as tangent vectors at points of  $M$ . A first promising replacement for transitivity is the following definition of plenitude: denote the sheaf of even  $\mathbb{Z}$ -degree increasing derivations on the sheaf of superfunctions  $\mathcal{O}_{\mathcal{M}}$  by  $Der_{\bar{0}}^{(2)}(\mathcal{O}_{\mathcal{M}})$ . A complex supermanifold  $\mathcal{M}$  is called  $2s$ -nildominated if the presheaf of common kernels of the  $[X, \cdot] \subset Der_{\bar{0}}^{(2)}(Der_{\bar{0}}^{(2)}(\mathcal{O}_{\mathcal{M}}))$  for  $X \in \mathcal{V}_{\mathcal{M},\bar{0}}^{(2)}$ , is contained in degree  $2s$  and higher in the  $\mathbb{Z}$ -filtration. We will prove the following result in section 2: fixing  $E \rightarrow M$ , the Čech equivalence class  $\alpha \in H^1(M, G_E)$  (see [Gr82]) of a  $2s$ -nildominated complex supermanifold  $\mathcal{M}$  is fixed up to errors of degree  $2s$  and higher by the explicit fields  $\mathcal{V}_{\mathcal{M},\bar{0}}^{(2)}$  regarded as a subset of  $C^0(M, Der_{\bar{0}}^{(2)}(\mathcal{O}_{\Lambda E}))$ .

In section 3 we discuss a graded version of nildominance naturally arising from the definition of nildominance. We prove that graded  $2s$ -nildominated supermanifolds are already split up to errors of degree  $2s$  and higher. So one might suspect that also  $2s$ -nildominated supermanifolds are split up to errors of degree  $2s$  and higher. A disappointing expectation. In section 4 we give an example of a  $4$ -nildominated supermanifold that is non-split being non-trivially deformed already in degree  $2$ . The example is constructed on  $\mathbb{P}^1(\mathbb{C})$  with odd dimension  $7$ . Although it might be suspected that many examples exist, it is technically difficult to construct one. Reasons are explained in the respective section.

Unfortunately the notion of  $2s$ -nildominance is not stable with respect to products of supermanifolds, a feature that holds for homogeneity of supermanifolds. Section 5 presents the final and stricter notion of plenitude of unipotent automorphisms. Here the presheaf of common kernels of the  $[X, \cdot] \subset Der_{\bar{0}}^{(2)}(Der(\mathcal{O}_{\mathcal{M}}))$  for  $X \in \mathcal{V}_{\mathcal{M},\bar{0}}^{(2)}$ , is asked to lie in degree  $t$  and higher. A complex supermanifold satisfying this will be called strictly  $t$ -nildominated. We prove that strict  $t$ -nildominance of the factors induces strict  $t$ -nildominance of the product of complex supermanifolds. Finally strict  $t$ -nildominance still allows non-split examples: the example of section 4 is shown to be strictly  $3$ -nildominated.

# 1 Notation

First we fix notation. We consider supermanifolds in the sense of Berezin, Kostant, and Leites. For details see e.g. [Ko77], [Le80], [DM99]. Let  $\mathcal{M}$  be a complex supermanifold with sheaf of superfunctions  $\mathcal{O}_{\mathcal{M}}$  containing the subsheaf  $\mathcal{N}_{\mathcal{M}}$  of nilpotent elements. Denote the induced  $\mathbb{Z}$ -filtration by upper indexes in brackets, so  $\mathcal{O}_{\mathcal{M}}^{(0)} = \mathcal{O}_{\mathcal{M}}$ ,  $\mathcal{O}_{\mathcal{M}}^{(1)} = \mathcal{N}_{\mathcal{M}}$  etc. and note that the sheaves of superderivations  $Der(\mathcal{O}_{\mathcal{M}})$  and of endomorphisms  $End(\mathcal{O}_{\mathcal{M}})$  inherit the  $\mathbb{Z}$ -filtration. Denote the underlying manifold of  $\mathcal{M}$  by  $M$  and the associated complex vector bundle defined via  $\mathcal{O}_{\mathcal{M}}^{(1)}/\mathcal{O}_{\mathcal{M}}^{(2)}$  by  $E \rightarrow M$ . The  $\mathbb{Z}$ -graded supermanifold associated to  $\mathcal{M}$  is  $gr(\mathcal{M}) = (M, \mathcal{O}_{\Lambda E})$  with  $\mathcal{O}_{\Lambda E}$  denoting the sheaf of sections in  $\Lambda E$ . The  $\mathbb{Z}$ -grading will be denoted by lower indexes. Let  $\alpha \in H^1(M, G_E)$  be an associated non-split cohomology class of  $\mathcal{M}$  (see [Gr82]). Here  $G_E$  is the sheaf of those (even) automorphisms of the sheaf of  $\mathbb{Z}/2\mathbb{Z}$ -graded algebras  $\mathcal{O}_{\Lambda E}$  that have the identity as degree zero part. Further  $H^1(M, G_E)$  denotes the first Čech cohomology induced by concatenation of morphisms. Now  $\alpha$  is well-defined up to the action of  $H^0(M, Aut(E))$ , the global automorphisms of  $E$ . Following [Ro85], it is  $\alpha = [\exp(u)]$  represented by a cochain  $u$  in  $C^1(M, Der_0^{(2)}(\mathcal{O}_{\Lambda E}))$ . The lower index with bar denotes the  $\mathbb{Z}/2\mathbb{Z}$ -grading. If  $u$  can be chosen to lie in  $C^1(M, Der_0^{(2s)}(\mathcal{O}_{\Lambda E}))$  then  $\mathcal{M}$  will be called *split up to errors of degree  $2s$  and higher*.

Let  $\{U_i\}$  be a Leray covering of coordinate charts of  $M$  with respect to coherent sheaf cohomology such that the  $U_{ij} := U_i \cap U_j$  are connected. Then superfunctions  $f \in \mathcal{O}_{\mathcal{M}}(U_i \cup U_j)$  with  $f_l = f|_{U_l} \in \mathcal{O}_{\Lambda E}|_{U_l}$ ,  $l = i, j$ , transform with  $f_i = \alpha_{ij}(f_j)$ . So for derivations  $X = Der(\mathcal{O}_{\mathcal{M}})(U_i \cup U_j)$  with  $X_l = X|_{U_l} \in Der(\mathcal{O}_{\Lambda E}|_{U_l})$ ,  $l = i, j$ , we have:

$$X_i = \alpha_{ij} X_j \alpha_{ij}^{-1} \quad (1)$$

Let  $\mathcal{V}_{\mathcal{M}} = H^0(M, Der(\mathcal{O}_{\mathcal{M}}))$  be the Lie superalgebras of global super vector fields on  $\mathcal{M}$  and  $\mathcal{V}_{\Lambda E} = H^0(M, Der(\mathcal{O}_{\Lambda E}))$ . Then  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  is the Lie algebra of the group of unipotent automorphisms of  $\mathcal{M}$ . By the above considerations we can regard  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  as a subset of  $C^0(M, Der_0^{(2)}(\mathcal{O}_{\Lambda E}))$ . Now (1) for  $X$  running through  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  yields conditions on  $u = \log(\alpha)$ . If these conditions determine  $u$  up to terms in  $C^1(M, Der_0^{(2s)}(\mathcal{O}_{\Lambda E}))$  then  $\mathcal{M}$  will be called *determined up to errors of degree  $2s$  and higher by its unipotent automorphisms*.

For an  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  set  $k(X)$  to be the maximal positive integer with  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(k(X))}$ . We obtain a linear map  $X_{\bullet} : \mathcal{O}_{\mathcal{M}}^{(t)}/\mathcal{O}_{\mathcal{M}}^{(t+1)} \rightarrow \mathcal{O}_{\mathcal{M}}^{(t+k(X))}/\mathcal{O}_{\mathcal{M}}^{(t+k(X)+1)}$  for any  $t \geq 0$ , and linearly continue it to the sheaf  $\mathcal{O}_{\Lambda E} = \bigoplus_{t \geq 0} \mathcal{O}_{\mathcal{M}}^{(t)}/\mathcal{O}_{\mathcal{M}}^{(t+1)}$ . The resulting object is a derivation  $X_{\bullet} \in H^0(M, Der_0^{k(X)}(\mathcal{O}_{\Lambda E}))$ . Note that  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)} \rightarrow H^0(M, Der_0^{(2)}(\mathcal{O}_{\Lambda E}))$ ,  $X \mapsto X_{\bullet}$  is in general not linear.

## 2 Nildominated supermanifolds

We give sense to the expression of plenitude of unipotent automorphisms.

**Definition 2.1.** A complex supermanifold  $\mathcal{M}$  is called  $2s$ -nildominated,  $s \in \mathbb{N}$ , if for any open set  $U \subset M$  we have:

$$\bigcap_{X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}} \text{Ker} \left( [X, \cdot] : \text{Der}_0^{(2)}(\mathcal{O}_{\mathcal{M}})(U) \rightarrow \text{Der}_0^{(2)}(\mathcal{O}_{\mathcal{M}})(U) \right) \subset \text{Der}_0^{(2s)}(\mathcal{O}_{\mathcal{M}})(U) \quad (2)$$

In particular  $2s$ -nildominance forces the non-trivial center of  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  to lie in  $\mathcal{V}_{\mathcal{M}, \bar{0}}^{(2s)}$ . If the odd dimension of  $\mathcal{M}$  is  $2s$  or  $2s + 1$  then  $\mathcal{M}$  can be at most  $2s$ -nildominated. In this case  $\mathcal{M}$  is simply called *nildominated*. We can follow:

**Proposition 2.2.** A  $2s$ -nildominated supermanifold  $\mathcal{M}$  is determined up to errors of degree  $2s$  and higher by its unipotent automorphisms. In particular a nildominated supermanifold  $\mathcal{M}$  with  $H^1(M, \text{Der}_0^{(rk(E)-1)}(\mathcal{O}_{\Lambda E})) = 0$  is determined by its unipotent automorphisms.

*Proof.* For  $s = 1$  the statement is trivial. Let  $s > 1$  and  $(i, j)$  with  $U_{ij} := U_i \cap U_j \neq \emptyset$ . Let  $(\alpha_{ij})_{ij} \in Z^1(M, G_E)$  represent  $\alpha$  and assume that  $(\gamma_{ij})_{ij} \in Z^1(M, G_E)$  is another cochain yielding a supermanifold as described in the proposition. Then there is a  $(\beta_{ij})_{ij} \in C^1(M, G_E)$  with  $\beta_{ij}\alpha_{ij} = \gamma_{ij}$  and  $\beta_{ij}X_i = X_i\beta_{ij}$  for all  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  due to (1). Let  $Y_{ij} = \log(\beta_{ij}) \in \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E})(U_{ij})$ . Now:

$$0 = [X_i, \beta_{ij}] = \sum_{n=0}^{\infty} \frac{1}{n!} [X_i, Y_{ij}^n] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{l=0}^{n-1} Y_{ij}^l [X_i, Y_{ij}] Y_{ij}^{n-l-1} \quad (3)$$

Decompose  $[X_i, Y_{ij}] = \sum_{t=2}^s Z_{2t}$ . Now for reasons of degree  $Z_4 = 0$  follows directly from (3). Assuming that  $Z_{2t} = 0$  for all  $t < u$  we find for reasons of degree the only contribution to the degree  $2u$  term in (3) is in the summand for  $n = 1$ . We have  $Z_{2u}$  hence being zero. So  $[X_i, Y_{ij}] = 0$  for all  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  and from  $2s$ -nildominance follows  $Y_{ij} \in \text{Der}_0^{(2s)}(\mathcal{O}_{\mathcal{M}})(U)$ .  $\square$

Finally we prove a local criterion for  $2s$ -nildominance for later application.

**Proposition 2.3.** A complex supermanifold  $\mathcal{M}$  is  $2s$ -nildominated if and only if (2) holds for an arbitrary polydisc  $U$  in each of the connected components of the underlying manifold  $M$ .

*Proof.* Assume  $rk(E) \geq 2$ . We introduce the integer valued function  $F : M \rightarrow \mathbb{N}$  by setting  $F(p)$  for  $p \in M$  to be the maximal integer  $s$  such that any open neighborhood  $V$  of  $p$  contains an open set  $U \subset V$  where (2) holds for  $s$ . Note that  $F$  is bounded by 1 and  $\lfloor \frac{1}{2}rk(E) \rfloor$  and hence well-defined.

Let  $m \in \mathbb{N}$ ,  $p_n \in M$  a sequence with limit  $p \in M$  and  $F(p_n) = m$  for all  $n$ . Further let  $V$  be an arbitrary neighborhood of  $p$ . Then there is an  $n_0$  such that  $p_{n_0} \in V$ . Further  $V$  is a neighborhood of  $p_{n_0}$  and hence contains an open  $U \subset V$  where (2) holds for  $m$ . So  $F(p) \geq F(p_{n_0})$ . Setting  $T := \max\{F(M)\}$  we have that  $F^{-1}(T)$  is closed.

Now let  $q \in F^{-1}(T)$  and let  $V_n$ ,  $n \in \mathbb{N}$ , be a base of neighborhoods of  $q$ . Let  $k_n \in \mathbb{N}$  be the maximal  $s$  such that (2) holds in  $V_n$  for  $s$ . Now  $k_n$  is decreasing and becomes stationary for

$n \geq n_0$  with value  $k$ . In particular on  $V_{n_0}$  there exists a  $Z \in \text{Der}_0^{(2k)}(\mathcal{O}_{\Lambda E})$  with  $[X, Z] = 0$  for all  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$ . So on any open set  $U \subset V_{n_0}$  the existence of  $Z|_U$  forces (2) to be false for  $k + 1$ . In particular  $F(q) \leq k$  and hence  $k = T$ . Again the existence of  $Z$  yields that  $F(V_{n_0}) = k = T$ . So  $F^{-1}(T)$  is open.

So  $F$  is constant on connected components. The result follows since a polydisc is Stein.  $\square$

In particular for a connected  $M$  the maximal  $s$  such that  $\mathcal{M}$  is (graded)  $2s$ -nildominated can be determined in any local coordinate chart as soon as the restricted global vector fields are known.

### 3 Graded nildominated supermanifolds

It is natural to consider a graded version of nildominance.

**Definition 3.1.** A complex supermanifold  $\mathcal{M}$  is called *graded  $2s$ -nildominated*,  $s \in \mathbb{N}$ , if for any open set  $U \subset M$  we have:

$$\bigcap_{X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}} \text{Ker}([X_\bullet, \cdot] : \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E})(U) \rightarrow \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E})(U)) \subset \text{Der}_0^{(2s)}(\mathcal{O}_{\Lambda E})(U) \quad (4)$$

Again if the odd dimension of  $\mathcal{M}$  is  $2s$  or  $2s + 1$  then  $\mathcal{M}$  is simply called *graded nildominated*. Assuming graded  $2s$ -nildominance, let  $Y \in \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E})(U)$ . Now  $[X, Y] = 0$  for all  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  includes  $[X_\bullet, Y_\bullet] = 0$  for all  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$ . Hence  $Y_\bullet \in \text{Der}_0^{(2s)}(\mathcal{O}_{\Lambda E})(U)$  and  $Y \in \text{Der}_0^{(2s)}(\mathcal{O}_{\mathcal{M}})(U)$ . So graded  $2s$ -nildominance is a stronger assumption than  $2s$ -nildominance. In fact it is too strong:

**Proposition 3.2.** A graded  $2s$ -nildominated supermanifold  $\mathcal{M}$  is split up to error of degree  $2s$  and higher. In particular a graded nildominated supermanifold  $\mathcal{M}$  satisfying the condition  $H^1(M, \text{Der}_0^{(rk(E)-1)}(\mathcal{O}_{\Lambda E})) = 0$  is split.

*Proof.* Let  $\alpha = \exp(Y) \in H^1(M, G_E)$  be the associated non-split class. We can follow from (1) that  $[X_\bullet, Y_\bullet] = 0$  for all  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$ . Hence  $Y \in C^1(M, \text{Der}_0^{(2s)}(\mathcal{O}_{\Lambda E}))$ . The cocycle condition for  $\alpha$  yields the second statement.  $\square$

An analog of Proposition 2.3 exists with similar arguments also for graded  $2s$ -nildominance. Further we obtain:

**Corollary 3.3.** Let  $\mathcal{M}$  be a nildominated supermanifold of odd dimension  $2s$  or  $2s + 1$ . If  $\mathcal{M}$  is graded  $(2s - 2)$ -nildominated then it is split up to errors of degree  $2s$ .

*Proof.* For any local derivation  $Y \in (\text{Der}_0^{(2s-2)}(\mathcal{O}_{\Lambda E}) \setminus \text{Der}_0^{(2s)}(\mathcal{O}_{\Lambda E}))(U)$  there is an  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  with  $[X, Y] \neq 0$ . Locally decompose  $X = X_2 + X_4 + \dots$  and  $Y = Y_{2s-2} + Y_{2s}$  according to the  $\mathbb{Z}$ -grading in coordinates. Reasons of degree yield  $[X_\bullet, Y] \neq 0$  so  $\mathcal{M}$  is graded  $2s$ -nildominated and Proposition 3.2 yields the statement.  $\square$

Proposition 3.2 produces the question, whether there exist  $2s$ -nildominated supermanifolds that is not split up to errors of degree  $2s$  and higher, and in particular not graded  $2s$ -nildominated. The answer is positive, an example is given in the section 4.

## 4 A 4-nildominated example, non-split in degree 2

Due to Corollary 3.3 there is no nildominated example of odd dimension smaller than 6 that is not split up to errors of highest even degree. Here we construct a 4-nildominated counter example in odd dimension 7 on  $M = \mathbb{P}^1(\mathbb{C})$  that is non-split deformed already in degree 2. We use the standard coordinate charts  $U_i = \{[z_0 : z_1] \mid z_i \neq 0\}$  for  $i = 0, 1$  and denote  $z = \frac{z_1}{z_0}$  and  $w = \frac{z_0}{z_1}$ . It should be mentioned that there are several conditions that an example of this kind has to satisfy reducing the hope for a technically easy example. First there have to be enough fields in  $\mathcal{V}_{\mathcal{M},0}^{(2)}$  to ensure 4-nildominance. Secondly there have to be not too many of them to prevent graded 4-nildominance which would include a splitting up to errors of degree 4 and higher. Then  $H^1(M, \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E}))$  has to contain candidates for deformations that allow the elements in  $\mathcal{V}_{\mathcal{M},0}^{(2)}$  to produce 4-nildominance from graded 2-nildominance. It turns out that  $E = 3TM^{\otimes 2} \oplus 4TM^*$  yields a good starting point. The deformation has to be chosen such that for some global vector fields  $X_i - X_{\bullet,i}$  is forced to be nontrivial on  $U_0$  and for some to be nontrivial on  $U_1$ . Otherwise it can be shown that 4-nildominance is not possible.

For a line bundle  $\mathcal{O}(k)$  we use bundle coordinates  $(z, \xi)$  and  $(w, \xi')$  on  $U_0$ , resp.  $U_1$ , transforming with  $w = \frac{1}{z}$  and  $\xi' = w^k \xi$ . In this notion  $TM \cong \mathcal{O}(2)$ . We use the induced identifications  $H^0(M, \mathcal{O}(k)) \cong \mathbb{C}[z]_{\leq k}$  and  $H^1(M, \mathcal{O}(k)) = \frac{1}{z}\mathbb{C}[\frac{1}{z}]_{\leq -2-k}$  with functions on  $U_0$ , resp.  $U_0 \cap U_1$ . Set  $E = 3\mathcal{O}(4) \oplus 4\mathcal{O}(-2)$ . On  $U_0$  denote the fiber coordinates on  $3\mathcal{O}(4)$  by  $\theta_1, \theta_2, \theta_3$ , and the fiber coordinates on  $4\mathcal{O}(-2)$  by  $\eta_1, \dots, \eta_4$ . Following Proposition 2.3 we will do the concrete calculations on  $U_0$ . By direct calculation we obtain for the global fields:

**Lemma 4.1.** *The elements in  $\mathcal{V}_{\Lambda E,0}^{(2)}$  are represented on  $U_0$  by sums of the terms in the following table. The indexes  $i, \dots, m$  cover all allowed entries skipping elements that vanish due to nilpotency of the odd variables.*

degree 2		degree 4	degree 6
$\mathbb{C}[z]_{\leq 4}\theta_i\eta_j\partial_z$	$\mathbb{C}[z]_{\leq 10}\theta_i\theta_j\partial_z$	$\mathbb{C}[z]_{\leq 0}\theta_i\eta_j\eta_k\eta_l\partial_z$ $\mathbb{C}[z]_{\leq 6}\theta_i\theta_j\eta_k\eta_l\partial_z$ $\mathbb{C}[z]_{\leq 12}\theta_1\theta_2\theta_3\eta_j\partial_z$	$\mathbb{C}[z]_{\leq 2}\theta_j\theta_k\eta_1\eta_2\eta_3\eta_4\partial_z$ $\mathbb{C}[z]_{\leq 8}\theta_1\theta_2\theta_3\eta_i\eta_j\eta_k\partial_z$
$\mathbb{C}[z]_{\leq 2}\theta_i\theta_j\eta_k\partial_{\theta_i}$	$\mathbb{C}[z]_{\leq 8}\theta_1\theta_2\theta_3\partial_{\theta_i}$	$\mathbb{C}[z]_{\leq 4}\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\theta_k}$	$\mathbb{C}[z]_{\leq 0}\theta_1\theta_2\theta_3\eta_1\eta_2\eta_3\eta_4\partial_{\theta_i}$
$\mathbb{C}[z]_{\leq 2}\theta_i\eta_j\eta_k\partial_{\eta_l}$	$\mathbb{C}[z]_{\leq 8}\theta_i\theta_j\eta_k\partial_{\eta_l}$	$\mathbb{C}[z]_{\leq 4}\theta_i\theta_j\eta_k\eta_l\eta_m\partial_{\eta_n}$	$\mathbb{C}[z]_{\leq 6}\theta_1\theta_2\theta_3\eta_1\eta_2\eta_3\eta_4\partial_{\eta_i}$
$\mathbb{C}[z]_{\leq 14}\theta_1\theta_2\theta_3\partial_{\eta_i}$		$\mathbb{C}[z]_{\leq 10}\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\eta_k}$	

The local common kernel in the sense of (2) as well as (4) on  $U_0$  of the global vector fields contains  $\theta_1\theta_2\theta_3\partial_{\eta_i}$ ,  $i = 1, \dots, 4$ .

Hence the constructed split supermanifold  $(M, \mathcal{O}_{\Lambda E})$  is 2-nildominated as well as graded 2-nildominated, the lowest regularity satisfied by any complex supermanifold. Now we deform the split supermanifold  $(M, \mathcal{O}_{\Lambda E})$  with

$$Y_{01} = \left( \frac{1}{z^2} \eta_1 + \frac{1}{z^8} \eta_2 \right) \eta_3 \eta_4 \partial_{\theta_1} \in H^1(M, \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E}))$$

yielding the cohomologically non-trivial:

$$\alpha_{01} = \exp(Y_{01}) = Id + \left( \frac{1}{z^2} \eta_1 + \frac{1}{z^8} \eta_2 \right) \eta_3 \eta_4 \partial_{\theta_1} \in H^1(M, G_E)$$

We denote the new non-split supermanifold by  $\mathcal{M}$ . First we regard the global vector fields with degree 2 part  $P_j(z) \theta_1 \eta_j \partial_z$ ,  $P_j(z) \in \mathbb{C}[z]_{\leq 4}$ ,  $j = 1, 2$ . For satisfying (1) we need to adjust the fields. Since higher degree terms vanish we obtain:

$$\begin{aligned} \alpha_{01}(P_j(z) \theta_1 \eta_j \partial_z) \alpha_{01}^{-1} &= P_j(z) \theta_1 \eta_j \partial_z + [Y_{01}, P_j(z) \theta_1 \eta_j \partial_z] \\ &= P_j(z) \theta_1 \eta_j \partial_z + \eta_1 \eta_2 \eta_3 \eta_4 \cdot \begin{cases} \left( -\frac{P_1(z)}{z^8} \partial_z + 8 \frac{P_1(z)}{z^9} \theta_1 \partial_{\theta_1} \right) & \text{for } j = 1 \\ \left( \frac{P_2(z)}{z^2} \partial_z - 2 \frac{P_2(z)}{z^3} \theta_1 \partial_{\theta_1} \right) & \text{for } j = 2 \end{cases} \end{aligned}$$

So in degree to obtain a global vector field we have to correct  $P_j(z) \theta_1 \eta_j \partial_z$  by adding degree 4 terms on  $U_0$ , resp.  $U_1$ . For  $j = 1$  we need  $P_1(z) \in \mathbb{C}[z]_{\leq 1}$  allowing corrections on  $U_1$ . For  $j = 2$  we need  $P_2(z) \in z^3 \mathbb{C}[z]_{\leq 1}$  allowing corrections on  $U_0$ .

**Lemma 4.2.** *Corrections of degree 2 fields exist if and only if correction terms of (pure) degree 4 exist. The following degree 2 vector fields yields global vector fields on  $\mathcal{M}$  and need not be corrected on  $U_0$  (but possibly on  $U_1$ ):*

vector field	condition	vector field	condition
$\mathbb{C}[z]_{\leq 4} \theta_i \eta_j \partial_z$	$j \notin \{1, 2\}$	$\mathbb{C}[z]_{\leq 0} \theta_i \eta_1 \eta_j \partial_{\eta_j}$	
$\mathbb{C}[z]_{\leq 1} \theta_i \eta_1 \partial_z$		$\mathbb{C}[z]_{\leq 8} \theta_i \theta_j \eta_k \partial_{\eta_l}$	$k \neq l \text{ and } k \in \{3, 4\}$
$\mathbb{C}[z]_{\leq 7} \theta_i \theta_j \partial_z$		$\mathbb{C}[z]_{\leq 0} \theta_i \theta_j \eta_k \partial_{\eta_k}$	$k \neq 2$
$\mathbb{C}[z]_{\leq 2} \theta_i \theta_j \eta_k \partial_{\theta_l}$	$1 \notin \{i, j\} \text{ or } k \notin \{1, 2\}$	$\mathbb{C}[z]_{\leq 6} \theta_i \theta_j \eta_2 \partial_{\eta_2}$	$1 \notin \{i, j\}$
$\mathbb{C}[z]_{\leq 0} \theta_1 \theta_j \eta_1 \partial_{\theta_l}$		$\mathbb{C}[z]_{\leq 0} \theta_1 \theta_i \eta_2 \partial_{\eta_2}$	
$\mathbb{C}[z]_{\leq 0} \theta_1 \theta_2 \theta_3 \partial_{\theta_i}$		$\mathbb{C}[z]_{\leq 6} \theta_i \theta_j \eta_1 \partial_{\eta_k}$	$k \neq 1$
$\mathbb{C}[z]_{\leq 2} \theta_i \eta_j \eta_k \partial_{\eta_l}$	$\{j, k\} \neq \{1, l\}$ and $\{j, k\} \neq \{2, l\}$	$\mathbb{C}[z]_{\leq 0} \theta_i \theta_j \eta_2 \partial_{\eta_k}$	$k \neq 2$
		$\mathbb{C}[z]_{\leq 6} \theta_1 \theta_2 \theta_3 \partial_{\eta_i}$	

The common kernel of  $[X, \cdot] : \text{Der}_4(\mathcal{O}_{\mathcal{M}})(U_0) \rightarrow \text{Der}_6(\mathcal{O}_{\mathcal{M}})(U_0)$  with  $X$  running through the list is the  $\mathcal{O}_{\mathcal{M}}(U_0)$ -module spanned by  $\eta_i \theta_1 \theta_2 \theta_3 \partial_z$ ,  $\theta_2 \theta_3 \eta_1 \eta_3 \eta_4 \partial_{\theta_1}$ , and  $\eta_i \eta_j \theta_1 \theta_2 \theta_3 \partial_{\eta_k}$ .

*Proof.* The first statement follows from  $Y_{01} D Y_{01} = Y_{01} Y_{01} = 0$  for any derivation  $D$ . The list only contains fields that either commute with  $Y_{01}$  or yield degree 4 terms that can be corrected by adding a degree 4 term on  $U_1$ . One inclusion of the last statement follows by direct calculation. For the converse inclusion assume that

$$Z = \sum_{IJ} f_{IJ} \theta^I \eta^J \partial_z + \sum_{IJK} g_{IJK} \theta^I \eta^J \partial_{\theta_k} + \sum_{IJK} h_{IJK} \theta^I \eta^J \partial_{\eta_k}$$

with numerical coefficient functions is in the common kernel. Now  $[\theta_1\theta_2\theta_3\partial_{\eta_1}, Z]$  contains  $f_{(000)(1111)}\theta^{(111)}\eta^{(0111)}\partial_z$ . Hence  $f_{(000)(1111)} = 0$ . Further  $[\theta_1\theta_2\eta_1\partial_{\eta_2}, Z]$  contains the summand  $f_{(001)(0111)}\theta^{(111)}\eta^{(1011)}\partial_z$ . So varying the indexes in  $\theta_i\theta_j\eta_k\partial_{\eta_l}$ , by analog arguments  $f_{IJ} = 0$  for  $|I| = 1$ . Further  $[\theta_1\eta_2\eta_3\partial_{\eta_4}, Z]$  contains  $f_{(011)(1001)}\theta^{(111)}\eta^{(1110)}\partial_z$ . So analogously varying the indexes of  $\theta_i\eta_j\eta_k\partial_{\eta_l}$  in the allowed range, we obtain  $f_{IJ} = 0$  for  $|I| = 2$ . In particular the first summand of  $Z$  is a  $\mathcal{O}_M(U_0)$ -linear combination of  $\eta_i\theta_1\theta_2\theta_3\partial_z$ . Now  $[\theta_1\theta_2\eta_1\partial_{\eta_1}, Z]$  contains  $\sum_k g_{(001)(1111)k}\theta^{(111)}\eta^{(1111)}\partial_{\theta_k}$ . Altogether  $g_{IJk} = 0$  for  $|I| = 1$  by varying  $i, j$  in  $\theta_i\theta_j\eta_1\partial_{\eta_1}$ . Further  $[\theta_1\eta_1\eta_2\partial_{\eta_2}, Z]$  contains  $\sum_k g_{(011)(0111)k}\theta^{(111)}\eta^{(1111)}\partial_{\theta_k}$ . Since  $\theta_i\eta_2\eta_j\partial_{\eta_j}$  is not allowed we obtain  $g_{IJk} = 0$  for  $|I| = 2$  and  $J \neq (1011)$ . Now  $[\theta_2\theta_3\eta_2\partial_{\theta_2}, Z]$  contains  $g_{(110)(1011)1}\theta^{(111)}\eta^{(1111)}\partial_{\theta_1}$  and  $g_{(110)(1011)3}\theta^{(111)}\eta^{(1111)}\partial_{\theta_3}$ . With  $\theta_2\theta_3\eta_2\partial_{\theta_3}$  it follows  $g_{IJk} = 0$  for  $|I| = 2$ ,  $J = (1011)$  and  $(I, k) \in \{((110), 1), ((110), 3), ((101), 1), ((101), 2)\}$ . Now  $[\theta_1\theta_2\partial_z, Z]$  contains  $(g_{(011)(1011)2} + g_{(101)(1011)1})\theta^{(111)}\eta^{(1011)}\partial_z$ . In particular  $g_{(011)(1011)2} = 0$ . Further  $[\theta_1\theta_3\partial_z, Z]$  contains  $(g_{(110)(1011)1} - g_{(011)(1011)3})\theta^{(111)}\eta^{(1011)}\partial_z$  hence  $g_{(011)(1011)3} = 0$ . We switch to the third summand of the derivation  $Z$ . The element  $[\theta_1\theta_2\theta_3\partial_{\theta_1}, Z]$  contains  $\sum_k h_{(100)(1111)k}\theta^{(111)}\eta^{(1111)}\partial_{\eta_k}$  so varying  $\theta_1\theta_2\theta_3\partial_{\theta_i}$  observe  $h_{IJk} = 0$  for  $|I| = 1$ . Now we can continue with the analysis of the second part of  $Z$ . The element  $[\theta_2\theta_3\eta_2\partial_{\theta_2}, Z]$  contains  $(g_{(110)(1011)2} - g_{(110)(1011)2} - g_{(101)(1011)3})\theta^{(111)}\eta^{(1111)}\partial_{\theta_2}$ . So  $g_{(101)(1011)3} = 0$ . Now  $[\theta_2\theta_3\partial_z, Z]$  contains  $(g_{(110)(1011)2} + g_{(101)(1011)3})\theta^{(111)}\eta^{(1011)}\partial_z$ . So  $g_{(110)(1011)2} = 0$  and  $g_{(011)(1011)1}$  is the only remaining  $g_{IJk}$  with  $|I| = 2$ . Now  $[\theta_1\theta_2\eta_3\partial_{\theta_1}, Z]$  contains  $\sum_k h_{(101)(1101)k}\theta^{(111)}\eta^{(1111)}\partial_{\eta_k}$ . Since  $1 \notin \{i, j\}$  in  $\theta_i\theta_j\eta_2\partial_{\theta_k}$  we can only conclude  $h_{IJk} = 0$  for  $|I| = 2$  and  $(I, J) \neq ((011), (1011))$ . Further  $[\theta_1\eta_1\eta_3\partial_{\eta_2}, Z]$  contains  $g_{(111), (0101), 1}\theta^{(111)}\eta^{(1111)}\partial_{\eta_2}$ . Varying  $\theta_i\eta_j\eta_k\partial_{\eta_l}$  with  $k \neq l$  we obtain  $g_{IJk} = 0$  for  $|I| = 3$ . Now  $[\theta_1\eta_1\eta_2\partial_{\eta_3}, Z]$  contains the term  $h_{(011)(1011)1}\theta^{(111)}\eta^{(1111)}\partial_{\eta_3}$ . Varying  $\theta_1\eta_j\eta_2\partial_{\eta_k}$  with  $j \neq k$  we finally obtain  $h_{IJk} = 0$  for  $|I| = 2$ .  $\square$

**Lemma 4.3.** *The following degree 4 vector fields yields global vector fields on  $\mathcal{M}$  and need not be corrected on  $U_0$  (but possibly on  $U_1$ ):*

vector field	condition	vector field	condition
$\mathbb{C}[z]_{\leq 0}\theta_i\eta_j\eta_k\eta_l\partial_z$	$j \in \{3, 4\}$	$\mathbb{C}[z]_{\leq 4}\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\theta_k}$	$\{i, j\} \neq \{1, k\} \text{ and } \{i, j\} \neq \{2, k\}$
$\mathbb{C}[z]_{\leq 6}\theta_i\theta_j\eta_k\eta_l\partial_z$		$\mathbb{C}[z]_{\leq 4}\theta_i\theta_j\eta_k\eta_l\eta_m\partial_{\eta_n}$	
$\mathbb{C}[z]_{\leq 12}\theta_1\theta_2\theta_3\eta_j\partial_z$		$\mathbb{C}[z]_{\leq 10}\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\eta_k}$	
$\mathbb{C}[z]_{\leq 9}\theta_1\theta_2\theta_3\eta_1\partial_z$		$\mathbb{C}[z]_{\leq 8}\theta_1\theta_2\theta_3\eta_1\eta_i\partial_{\eta_i}$	
$\mathbb{C}[z]_{\leq 3}\theta_1\theta_2\theta_3\eta_2\partial_z$		$\mathbb{C}[z]_{\leq 2}\theta_1\theta_2\theta_3\eta_2\eta_i\partial_{\eta_i}$	

The common kernel of  $[X, \cdot] : \text{Der}_2(\mathcal{O}_{\mathcal{M}})(U_0) \rightarrow \text{Der}^{(4)}(\mathcal{O}_{\mathcal{M}})(U_0)$  with  $X$  running through this list and through the list of Lemma 4.2 is the  $\mathcal{O}_M(U_0)$ -module spanned by  $\theta_1\theta_2\theta_3\partial_{\eta_i}$ .

*Proof.* The list only contains fields that either commute with  $Y_{01}$  or yield degree 6 terms that can be corrected by adding a degree 6 term on  $U_1$ . One inclusion of the last statement is direct calculation. The converse inclusion can be seen as follows. Let

$$Z = \sum_{IJ} f_{IJ}\theta^I\eta^J\partial_z + \sum_{IJk} g_{IJk}\theta^I\eta^J\partial_{\theta_k} + \sum_{IJk} h_{IJk}\theta^I\eta^J\partial_{\eta_k}$$



with numerical coefficient functions be an element of the common kernel. Now the element  $[\theta_1\theta_2\theta_3\eta_1\eta_2\partial_{\eta_3}, Z]$  contains  $f_{(000)(0011)}\theta^{(111)}\eta^{(1101)}\partial_z$ . Varying  $\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\eta_k}$  with  $k \notin \{i, j\}$  yields  $f_{IJ} = 0$  for  $|I| = 0$ . And  $[\theta_1\theta_2\eta_1\eta_2\eta_3\partial_{\eta_3}, Z]$  contains  $f_{(001)(0010)}\theta^{(111)}\eta^{(1110)}\partial_z$  so varying  $\theta_i\theta_j\eta_k\eta_l\eta_m\partial_{\eta_n}$  we obtain  $f_{IJ} = 0$  for  $|I| = 1$ . Further  $[\theta_1\theta_2\eta_3\partial_{\eta_2}, Z]$  contains  $f_{(011)(0000)}\theta^{(111)}\eta^{(0010)}\partial_z$ . Variations of  $\theta_i\theta_j\eta_3\partial_{\eta_j}$  yield  $f_{IJ} = 0$  for  $|I| = 2$ . Further  $[\theta_1\theta_2\theta_3\eta_1\eta_2\partial_{\eta_2}, Z]$  contains  $\sum_k g_{(000)(0111)k}\theta^{(111)}\eta^{(1111)}\partial_{\theta_k}$ . Varying  $\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\eta_j}$  we have  $g_{IJK} = 0$  for  $|I| = 0$ . Now  $[\theta_1\theta_2\eta_1\eta_2\eta_3\partial_{\eta_3}, Z]$  contains  $\sum_k g_{(001)(0011)k}\theta^{(111)}\eta^{(1111)}\partial_{\theta_k}$  and varying  $\theta_i\theta_j\eta_k\eta_l\eta_m\partial_{\eta_n}$  we obtain  $g_{IJK} = 0$  for  $|I| = 1$ . And  $[\theta_1\eta_1\eta_2\partial_{\eta_3}, Z]$  contains  $\sum_k g_{(011)(0010)k}\theta^{(111)}\eta^{(1100)}\partial_{\theta_k}$ . Variations of  $\theta_i\eta_j\eta_k\partial_{\eta_l}$ ,  $l \notin \{j, k\}$  yield  $g_{IJK} = 0$  for  $|I| = 2$ . Now  $[\theta_1\theta_2\theta_3\eta_1\partial_z, Z]$  contains  $\sum_{|J|=3} h_{(000)J1}\theta^{(111)}\eta^J\partial_z$ . Varying  $\theta_1\theta_2\theta_3\eta_i\partial_z$  we have  $h_{IJK} = 0$  for  $|I| = 0$ . Further  $[\theta_1\theta_2\theta_3\eta_1\eta_2\partial_{\theta_1}, Z]$  contains  $\sum_k h_{(100)(0011)k}\theta^{(111)}\eta^{(1111)}\partial_{\eta_k}$ . Varying  $\theta_1\theta_2\theta_3\eta_i\eta_j\partial_{\theta_k}$  we have  $h_{IJK} = 0$  for  $|I| = 1$ . Now  $[\theta_1\theta_2\eta_3\partial_{\theta_1}, Z]$  contains  $\sum_l (h_{(101)(1000)l}\eta_1 + h_{(101)(0100)l}\eta_2 + h_{(101)(0001)l}\eta_4)\eta_3\theta^{(111)}\partial_{\eta_l}$ . Varying  $\theta_i\theta_j\eta_k\partial_{\theta_j}$ ,  $k \in \{3, 4\}$  we find  $h_{IJK} = 0$  for  $|I| = 2$ . Further  $[\theta_1\eta_1\eta_2\eta_3\partial_z, Z]$  contains  $g_{(111)(0000)1}\theta^{(111)}\eta^{(1110)}\partial_z$ . With  $\theta_i\eta_4\partial_z$  we obtain  $g_{IJK} = 0$  for  $|I| = 3$ .  $\square$

Assume that some local vector field  $V \in \text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E}) \setminus \text{Der}_0^{(4)}(\mathcal{O}_{\Lambda E})$  is in the common kernel of the  $[X, \cdot] : \text{Der}_0^{(2)}(\mathcal{O}_{\mathcal{M}})(U_0) \rightarrow \text{Der}_0^{(4)}(\mathcal{O}_{\mathcal{M}})(U_0)$ ,  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$ . Let  $V = V_2 + V_4 + V_6$  according to the  $\mathbb{Z}$ -grading. For global fields  $X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  of pure degree on  $U_0$  as they are displayed in Lemmas 4.2 and 4.3, we have in particular the condition  $[X, V_{2k}] = 0$  for  $k = 1, 2$ . So by Lemma 4.3, the only interesting case is  $V_2 = \sum_{i=1}^4 f_i \theta_1 \theta_2 \theta_3 \partial_{\eta_i}$  with  $f_i \in \mathcal{O}_M(U_0)$ . So on the one hand,  $V_4$  is forced to lie in the kernel described in Lemma 4.2.

Recall that the correction terms for global fields of degree 2 only appear in degree 4, see Lemma 4.2. Hence we can assume for all of these  $X = X_2 + X_4 \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}$  that  $[X_2, V_2] = 0$  and  $[X_2, V_4] = -[X_4, V_2]$ . Regarding the following corrected global vector fields on  $U_0$ :

$$X'^j := z^2 \theta_1 \eta_2 \eta_j \partial_{\eta_j} + \theta_1 \eta_1 \eta_2 \eta_3 \eta_4 \partial_{\theta_1}, \quad j \in \{3, 4\}$$

we have:

$$-[X_4'^j, V_2] = -\sum_{i=1}^4 f_i \theta_1 \theta_2 \theta_3 \eta_1 \eta_2 \eta_3 \eta_4 \partial_{\eta_i} = [z^2 \theta_1 \eta_2 \eta_j \partial_{\eta_j}, V_4]$$

So on the other hand, for  $i \neq j$  the summand  $f_i \theta_1 \theta_2 \theta_3 \eta_1 \eta_2 \eta_3 \eta_4 \partial_{\eta_i}$  can only be created by  $\frac{f_i}{z^2} \theta_2 \theta_3 \eta_1 \eta_3 \eta_4 \partial_{\eta_i}$  as a summand of  $V_4$ . Now Lemma 4.2 yields  $f_i = 0$  for all  $i$  and hence  $V_2 = 0$  contradicting the above assumption. We can directly follow:

**Proposition 4.4.** *The complex supermanifold  $\mathcal{M}$  is graded 2-nildominated but not graded 4-nildominated. At the same time  $\mathcal{M}$  is 4-nildominated. Further it is non-split being deformed already in degree 2.*

## 5 Strictly nildominated supermanifolds

Nil dominance is not stable under products in the category of complex supermanifolds. We modify our approach.

**Definition 5.1.** *A complex supermanifold  $\mathcal{M}$  is called strictly  $t$ -nildominated,  $t \in \mathbb{N}$ , if for any open set  $U \subset M$  we have:*

$$\bigcap_{X \in \mathcal{V}_{\mathcal{M}, \bar{0}}^{(2)}} \text{Ker}([X, \cdot] : \text{Der}(\mathcal{O}_{\mathcal{M}})(U) \rightarrow \text{Der}(\mathcal{O}_{\mathcal{M}})(U)) = \text{Der}^{(t)}(\mathcal{O}_{\mathcal{M}})(U) \quad (5)$$

If its odd dimension is  $t + 1$  then it is called strictly *nildominated*. In particular any strictly  $(2s - 1)$ - or strictly  $2s$ -nildominated supermanifold is  $2s$ -nildominated and following Proposition 2.2 it is determined up to errors of degree  $2s$  and higher by its unipotent automorphisms.

**Proposition 5.2.** *Let  $\mathcal{M}'$  and  $\mathcal{M}''$  be  $t'$ -, resp.  $t''$ -nildominated supermanifolds associated with the bundles  $E' \rightarrow M'$ , resp.  $E'' \rightarrow M''$ . Let  $\alpha' \in H^1(M', G_{E'})$  and  $\alpha'' \in H^1(M'', G_{E''})$  be the associated classes. Set  $E := E' \oplus E'' \rightarrow M' \times M'' =: M$  and  $t = \min\{t', t''\}$ . There is a supermanifold  $\mathcal{M}$  on  $E \rightarrow M$  with morphisms  $\mathcal{M} \rightarrow \mathcal{M}'$  and  $\mathcal{M} \rightarrow \mathcal{M}''$  reducing to the projections on the product of vector bundles. Further  $\mathcal{M}$  is strictly  $t$ -nildominated and well-defined up to errors in  $C^1(M, \text{End}_0^{(t)}(\mathcal{O}_{\Lambda E}))$  by these properties.*

*Proof.* Existence is given by  $\exp(\log(\alpha') + \log(\alpha''))$ . Let  $(z'_i, \xi'_\sigma), (z''_j, \xi''_\rho)$  be local coordinates for  $E' \rightarrow M'$ , resp.  $E'' \rightarrow M''$ . Further let  $Y = \sum_I \xi'^I Y'_I + \sum_J \xi''^J Y''_J$  be a general local derivation in  $\text{Der}_0^{(2)}(\mathcal{O}_{\Lambda E})$  with  $Y'_I \in \text{Der}(\mathcal{O}_{\Lambda E'})$  and  $Y''_J \in \text{Der}(\mathcal{O}_{\Lambda E''})$ . For  $X' \in \mathcal{V}_{\mathcal{M}', \bar{0}}^{(2)}$  we have  $[X', Y] = \sum_I \xi'^I [X', Y'_I] + \sum_J X'(\xi'^J) Y''_J$ . Assuming  $[X', Y] = 0$  for all  $X' \in \mathcal{V}_{\mathcal{M}', \bar{0}}^{(2)}$  we find by strict  $t'$ -nil dominance of  $\mathcal{M}'$  that  $Y'_I \in \text{Der}^{(t')}(\mathcal{O}_{\Lambda E'})$ . Analogously assuming  $[X'', Y] = 0$  for all  $X'' \in \mathcal{V}_{\mathcal{M}'', \bar{0}}^{(2)}$  yields  $Y''_J \in \text{Der}^{(t'')}(\mathcal{O}_{\Lambda E''})$ . Hence:

$$Y \in \text{Der}^{(t')}(\mathcal{O}_{\Lambda E'}) \oplus \text{Der}^{(t'')}(\mathcal{O}_{\Lambda E''}) \oplus \text{End}^{(t)}(\mathcal{O}_{\Lambda E}) \quad (6)$$

Setting  $Y = Y_{ij}$  we proceed as in the proof of Proposition 2.2 and obtain that  $\mathcal{M}$  is well-defined up to terms in  $C^1(M, \text{End}_0^{(t)}(\mathcal{O}_{\Lambda E}))$ . Further (5) follows from (6).  $\square$

**Corollary 5.3.** *The product of two strictly  $t$ -nildominated supermanifolds is strictly  $t$ -nildominated.*

An analog of Proposition 2.3 exists with similar arguments also for strict  $2s$ -nil dominance. Using this we return to the above example:

**Proposition 5.4.** *The complex supermanifolds  $\mathcal{M}$  in section 4 is strictly 3-nil dominated and non-split being deformed already in degree 2.*

*Proof.* Regard the common kernel of  $[X, \cdot] : \text{Der}_q(\mathcal{O}_{\mathcal{M}})(U_0) \rightarrow \text{Der}(\mathcal{O}_{\mathcal{M}})(U_0)$  for  $X$  in the lists of Lemmas 4.2 and 4.3 and further of the global degree 6 vector fields in Lemma 4.1. For  $q = -1$  let  $Z = \sum_k g_k \partial_{\theta_k} + \sum_k h_k \partial_{\eta_k}$  be in the common kernel and regard the commutators with the fields  $\theta_1 \theta_2 \theta_3 \partial_{\theta_i}$  to see that the  $g_k$  vanish. The commutators with  $\theta_1 \theta_2 \eta_i \partial_{\eta_3}$  then show that the  $h_k$  vanish. For  $q = 0$  let  $Z = f \partial_z + \sum_{ij} g_{ij} \theta_i \partial_{\theta_j} + \sum_{ij} \hat{g}_{ij} \eta_i \partial_{\eta_j} + \sum_{ij} \hat{h}_{ij} \theta_i \partial_{\eta_j} + \sum_{ij} h_{ij} \eta_i \partial_{\eta_j}$  be in the common kernel. Then  $[\theta_1 \theta_2 \theta_3 \eta_1 \eta_2 \eta_3 \eta_4 \partial_{\theta_i}, Z]$  and  $[\theta_1 \theta_2 \theta_3 \eta_1 \eta_2 \eta_3 \eta_4 \partial_{\eta_i}, Z]$  yield  $\hat{h}_{ij} = \hat{g}_{ij} = 0$  for all  $i, j$  and  $g_{ij} = h_{ij} = 0$  for  $i \neq j$ . Further the  $\partial_{\theta_k}$ , resp.  $\partial_{\eta_k}$ , terms in  $[\theta_i \theta_j \eta_1 \eta_2 \eta_3 \eta_4 \partial_z, Z]$ , resp.  $[\theta_1 \theta_2 \theta_3 \eta_i \eta_j \eta_k \partial_z, Z]$ , yield  $g_{ii}, h_{ii} \in \mathbb{C}$ . Further the coefficients of  $\partial_z$  in  $[\theta_i \theta_j \partial_z, Z] = 0$  yield  $f' = g_{ii} + g_{jj}$  for all  $i \neq j$ . In particular  $g_{ii} = g_{jj}$  for all  $i, j$  and  $f' = 2g_{ii}$ . Now  $0 = [z \theta_i \theta_j \partial_z, Z] = z[\theta_i \theta_j \partial_z, Z] - Z(z) \theta_i \theta_j \partial_z = Z(z) \theta_i \theta_j \partial_z$  and  $Z(z) = f$  yield in addition  $f = 0$  and so  $g_{ii} = 0$  for all  $i$ . From  $[\theta_2 \theta_3 \eta_i \partial_{\theta_1}, Z]$  we obtain  $h_{ii} = 0$  for all  $i$ . For  $q = 1$  let

$$Z = \sum_{IJ} f_{IJ} \theta^I \eta^J \partial_z + \sum_{IJk} g_{IJk} \theta^I \eta^J \partial_{\theta_k} + \sum_{IJk} h_{IJk} \theta^I \eta^J \partial_{\eta_k}$$

be in the common kernel. The  $\partial_z$  terms in  $[\theta_1 \theta_2 \theta_3 \partial_{\theta_i}, Z]$  and  $[\theta_2 \theta_3 \eta_i \partial_{\eta_i}, Z]$  yield  $f_{IJ} = 0$  for all indexes. The  $\partial_{\eta_k}$  terms in  $[\theta_1 \theta_2 \theta_3 \partial_{\theta_i}, Z]$  show  $h_{IJk} = 0$  for  $|I| = |J| = 1$ . The  $\partial_{\theta_k}$  terms in  $[\theta_i \theta_j \eta_l \partial_{\eta_l}, Z]$  yield  $g_{IJk} = 0$  for  $|I| = |J| = 1$  and  $|I| = 0$ . The  $\partial_{\eta_k}$  terms with two  $\theta$ s in  $[\theta_i \eta_j \eta_k \partial_{\eta_l}, Z]$  with  $l \notin \{j, k\}$  yield  $g_{IJk} = 0$  for  $|I| = 2$ . The  $\partial_z$  terms in  $[\theta_i \eta_j \partial_z, Z]$  yield  $h_{IJk} = 0$  for  $k \neq 2$  and  $|I| = 0$  or  $2$ . The  $\partial_z$  term in  $[\theta_i \eta_2 \eta_3 \eta_4 \partial_z, Z]$  yields  $h_{IJ2} = 0$  for  $|I| = 2$  and the  $\partial_z$  term in  $[\theta_1 \theta_2 \theta_3 \eta_2 \partial_z, Z]$  yields  $h_{IJ2} = 0$  for  $|I| = 0$ . For  $q = 2$  the result follows from 4-nildominance of  $\mathcal{M}$ .  $\square$

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